Fundamentals of MHD

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week 1 and week 4

Plan

1. A short history of MHD
2. Governing equations
3. Energetics
4. Dimensionless parameters
5. MHD approximations
6. MHD physical effects
7. Stability of MHD flows
8. MHD turbulence
9. Measurement techniques

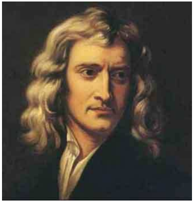
A short history of MHD

- The hydrodynamics chain
- The electromagnetism chain
- A short history of magnetohydrodynamics

The hydrodynamics chain

Newton (1642–1727) *Principia mathematica philosophiæ naturalis*, 1687

$$\mathbf{f} = m\mathbf{a}$$



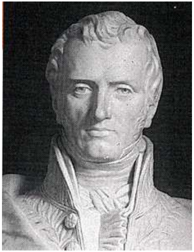
Euler (1701–1783) *Mémoires de l'académie des sciences de Berlin*, 1757

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

The hydrodynamics chain

Navier (1785–1836) French Engineer. Navier-Stokes equation, 1821



$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \rho\nu\nabla^2\mathbf{u}$$

Stokes (1819–1903) *On the theories of the internal friction of fluids in motion*, 1845

The electromagnetism chain

Volta (1745–1827) "voltaic pile" or battery, letter to the Royal Society, 1800

Ampère (1775–1836) unification of electricity and magnetism, 1820

The electromagnetism chain

Ohm (1789–1854) *Die galvanische Kette, mathematisch bearbeitet*, 1827

$$\mathbf{j} = \sigma \mathbf{E}$$

Faraday (1791–1867) the law of induction, 1831

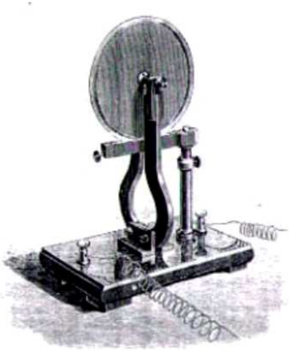
$$\Delta V = -\frac{d\Phi}{dt}$$

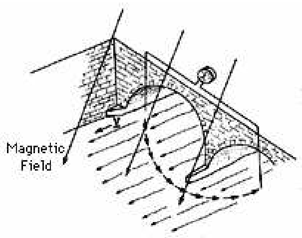
The electromagnetism chain

Maxwell (1831–1879) synthesis of electromagnetism "On Faraday's lines of force", 1855-1856

A short history of magnetohydrodynamics

Faraday (1791–1867) the dynamo disk





Faraday's six principles

- Have a little pad and take notes at all times
- Exchange letters with other scientists
- Have collaborations
- Check everything
- Avoid controversy
- Never make general assumptions too quickly, speak and write as precisely as possible

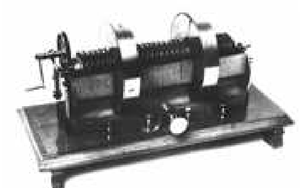
A short history of magnetohydrodynamics

Siemens (1816–1892) self-excited dynamo, 1866

A short history of magnetohydrodynamics

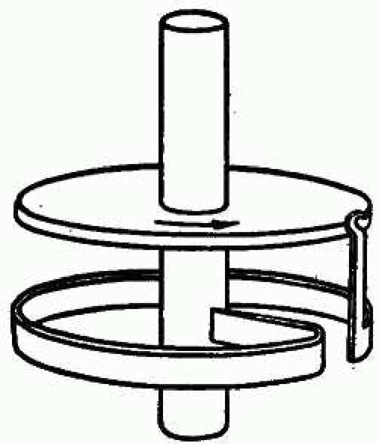
Siemens (1816–1892) self-excited dynamo, 1866

Ànyos Jedlik (1800–1895) first (?) self-excited dynamo, 1861



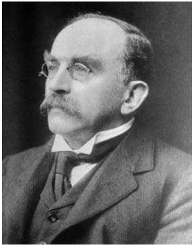
A short history of magnetohydrodynamics

Self-excited Faraday disk



A short history of magnetohydrodynamics

Larmor (1857–1942), dynamo action for the Sun and the Earth, 1919



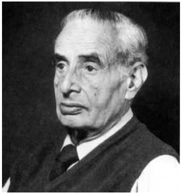
Hartmann stabilizing effect of externally imposed magnetic fields, 1937

A short history of magnetohydrodynamics

Alfvén (1908–1995) the mechanism of Alfvén waves, 1942

Alfvén wave demonstration Lundquist (1949), Lehnert (1953) and Jameson (1964) in liquid sodium. Bostik and Levine (1952), Allen et al. (1959), De Silva (1961) and Spillman (1963) in plasmas

Elsasser (1904–1991) father of Earth's dynamo magnetism



A short history of magnetohydrodynamics

Shercliff (1927–1983) structure of flows under an imposed magnetic field

A short history of magnetohydrodynamics

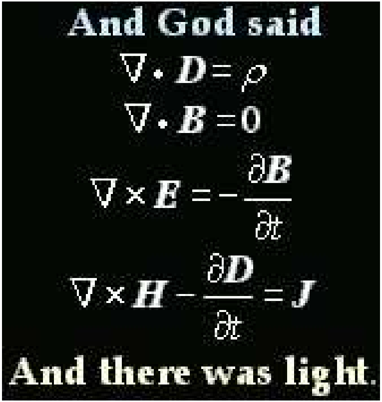
Kulikovskii (1933–) "characteristic surfaces", 1971

$$\int \frac{dl}{B} = C^{st}$$

A short history of magnetohydrodynamics

Demonstration of dynamo action Lowes and Wilkinson (1963) in an homogeneous solid with rotating cylinders. Gailitis (1999), Mühler (1999) and VKS team (2007) with liquid sodium

Governing equations



Governing equations

Navier-Stokes

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}.\nabla)\mathbf{u} = -\nabla p + \rho\mathbf{g} + \textit{Lorentz force} + \nabla\cdot[\rho\nu\nabla\mathbf{u}]$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

Maxwell

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= 0\\
\nabla\cdot\mathbf{E} &= q/\epsilon\\
\nabla\times\mathbf{B} &= \mu\mathbf{j} + \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}\\
\nabla\times\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}\\
\frac{\partial q}{\partial t} + \nabla\cdot\mathbf{j} &= 0
\end{aligned}$$

Ohm's law

Ohm's law

Correct Ohm's law (in a material reference system)

$$\mathbf{j} = \sigma \mathbf{E}'$$

Non-relativistic change of coordinates (G. Rousseaux, EuroPhys. Lett. **71**, 2005)

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B} \qquad \text{and} \qquad \mathbf{B}' = \mathbf{B}$$

General Ohm's law (with Hall effect)

$$\mathbf{j} = q\mathbf{u} + \sigma\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) + \frac{\omega_\tau}{|\mathbf{B}|}\mathbf{j} \times \mathbf{B}$$

Lorentz force

$$q\mathbf{E} + \mathbf{j} \times \mathbf{B}$$

Electrical charge equation

$$\frac{\epsilon}{\sigma}\left[\frac{\partial q}{\partial t} + (\mathbf{u}\cdot\nabla)q\right] + q = -\epsilon\nabla\cdot(\mathbf{u}\times\mathbf{B})$$

When ϵ/σ is much smaller than any timescale of the flow, then

$$q = -\epsilon\nabla\cdot(\mathbf{u}\times\mathbf{B})$$

It can then be seen that $q\mathbf{u}$ is negligible in Ohm's law. Similarly, $q\mathbf{E}$ is negligible in the Lorentz force and the displacement current $\epsilon\partial\mathbf{E}/\partial t$ is negligible in Maxwell's equations.

This is the so-called magneto-static approximation.

Magneto-static approximation $\epsilon/\sigma << \tau$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\nabla \cdot \mathbf{j} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{E} = q/\epsilon$$

$$\nabla \times \mathbf{B} = \mu \mathbf{j}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$q = -\epsilon \nabla \cdot (\mathbf{u} \times \mathbf{B})$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \rho \mathbf{g} + \mathbf{j} \times \mathbf{B} + \nabla \cdot [\rho \nu \nabla \mathbf{u}]$$

Magneto-static approximation $\epsilon/\sigma << \tau$

Taking the curl of Ohm's law

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \nabla \times [\eta \nabla \times \mathbf{B}]$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \rho \mathbf{g} + \frac{1}{\mu}[\nabla \times \mathbf{B}] \times \mathbf{B} + \nabla \cdot [\rho \nu \nabla \mathbf{u}]$$

where $\eta = 1/(\mu\sigma)$ is the magnetic diffusivity ($\mathrm{m^2 s^{-1}}$).

End of lecture 1